Department of Civil and Environmental Engineering and Geodetic Science
Geodetic and Geoinformation Science Section

GS521 Geodetic Control Surveying

The Transverse Mercator Projection

Introduction

The Transverse Mercator projection is another conformal projection that is the basis for SPCS zones in quite a number of states. The SPCS 27 TM zones use US Survey feet and are based on the Clarke 1866 ellipsoid, while those of SPCS 83 use meters and the GRS 80 ellipsoid, which is functionally the same as the WGS-84 ellipsoid and datum.

The formulae for conversion between geographical co-ordinates and grid co-ordinates are rather more long-winded than those for Lambert's Conformal Conic, but are more direct. Once you have set up such a conversion in a program or a spreadsheet, it will work well.

Again, the original tables developed for SPCS 27 using desktop calculators are not as good as these formulae, but the differences are fairly small. Redfearn's Formulae are generally the best and most rigorous for the conversion and are given below. They can as well be used for UTM zones as SPCS ones; all you have to do is allow for a different k_0 and, if using SPCS 1927, ellipsoid.

Remember that the Eastings will have a false origin value added, as will the Northings in the SPCS zones (and UTM in the southern hemisphere). The central meridian (λ_0) for the SPCS zones can be found in the tables, while it can be calculated for UTM zones. The origin latitude is given the Y or N co-ordinate of zero in the SPCS zones and the true N for this location on the central meridian gives the false origin offset for Y or N. The central meridian also receives a false origin offset, generally 2,000,000 ft in SPCS 27, but a variable number of meters for the SPCS 83 zones. For UTM, it is 500,000 meters. Check the tables handed out for the SPCS.

The central meridian scale factor, k_0, is 0·9996 for all UTM zones, but varies for the different SPCS zones. The value given in the attached tables is in the form 1 : x, where x is some value. To get the correct value of k_0, convert the 1 : x value to a decimal and subtract it from one, i.e.

$$k_0 = 1 - \frac{1}{x}$$

Geographicals to Grid Conversion

Given that we have a, e^2, ϕ, k_0, λ and λ_0, we can use the following expressions for the conversion. These are Redfearn's Formulae. Note that these use an extra term in the computations of E' and N', compared to Snyder's book, but this will make only a small difference in the overall values. The results will be a little different to the tabulated values for SPCS, too, owing to the limitations on the SPCS 27 computations. Remembering that the allowable distortion in the SPCS was to be no more than 1 in 10,000, it is acceptable to drop the final term in the formulae, as this doesn't degrade the formulae by anywhere near 1 in 10,000. These formulae will then agree with Snyder's formulae.

For UTM computations, you should use the full number of terms. This is because there is no 'legal' tolerance of distortion in the conversion process. UTM co-ordinates are now printed on 1:24,000 quadrangle maps, with either a grid/graticule or marginal ticks. These UTM co-ordinates are on the NAD27 datum and need to be converted to NAD83 before they can be used. While there is a marginal note concerning the conversion of latitude and longitude from NAD27 to NAD83 on many of the more recent mapsheets, this value **does not** apply to the UTM co-ordinates (or the SPCS co-ordinates). This is because the latitude and longitude values are, in effect, figured from the origin in Kansas, while the UTM Northing co-ordinates are figured from the Equator. SPCS northings are figured from the zone origin, so will have a different shift for each zone. You should convert the co-ordinates to latitude and longitude for the appropriate system, convert these to NAD83 (using a program such as that discussed below, or the spreadsheet on the server), then convert to UTM or SPCS TM co-ordinates. An approximate set of shifts for UTM can be found in a paper by Welch, R., and Homsey, A., "Datum Shifts for UTM Co-ordinates", in the *Photogrammetric Engineering and Remote Sensing* journal, Volume 63, No. 4, pp. 371–375, published in 1997.

Conversion Formulae

Easting

$$
\begin{aligned}
E' = k_0 \{ & \nu \omega \cos\phi \\
& + \nu \frac{\omega^3}{6} \cos^3\phi\, (\psi - t^2) \\
& + \nu \frac{\omega^5}{120} \cos^5\phi\, [4\psi^3 (1 - 6t^2) + \psi^2 (1 + 8t^2) - \psi (2t^2) + t^4] \\
& + \nu \frac{\omega^7}{5040} \cos^7\phi\, (61 - 479t^2 + 179t^4 - t^6) \}
\end{aligned}
$$

Northing

$$
\begin{aligned}
N' = k_0 \{ m & + \nu \sin\phi \frac{\omega^2}{2} \cos\phi \\
& + \nu \sin\phi \frac{\omega^4}{24} \cos^3\phi\, (4\psi^2 + \psi - t^2) \\
& + \nu \sin\phi \frac{\omega^6}{720} \cos^5\phi\, [8\psi^4(11-24t^2) - 28\psi^3(1-6t^2) + \psi^2(1-32t^2) - 2\psi t^2 \\
& \quad + t^4] \\
& + \nu \sin\phi \frac{\omega^8}{40320} \cos^7\phi\, (1385 - 3111t^2 + 543t^4 - t^6) \}
\end{aligned}
$$

Grid Convergence (in radians)

$$
\begin{aligned}
\gamma = & - \sin\phi\ \omega \\
& - \sin\phi \frac{\omega^3}{3} \cos^2\phi\, (2\psi^2 - \psi) \\
& - \sin\phi \frac{\omega^5}{15} \cos^4\phi\, [\psi^4 (11 - 24t^2) - \psi^3 (11 - 36t^2) + 2\psi^2 (1 - 7t^2) + \psi t^2] \\
& - \sin\phi \frac{\omega^7}{315} \cos^6\phi\, (17 - 26t^2 + 2t^4)
\end{aligned}
$$

Point Scale Factor

$$k = k_0 \{1 + \frac{\omega^2}{2} \cos^2 \phi \ \psi$$
$$+ \frac{\omega^4}{24} \cos^4 \phi \ [4 \psi^3 (1 - 6 t^2) + \psi^2 (1 + 24 t^2) - 4 \psi t^2]$$
$$+ \frac{\omega^6}{720} \cos^6 \phi \ (61 - 148 t^2 + 16 t^4) \ \}$$

where

$E' = E - E_0$ (E_0 is the offset of the central meridian; check the value for each zone. For UTM, E_0 = 500 000·000 meters.)

$N' = N - N_0$ (N_0 is the offset of the origin latitude; check the value for each zone. For UTM in the northern hemisphere, $N_0 = 0$; for UTM in the southern hemisphere, N_0 = 10 000 000·000 meters.)

ν = radius of curvature in the prime vertical at ϕ; i.e. $\nu = \frac{a}{\sqrt{1 - e^2 \sin^2 \phi}}$

$\rho = \frac{a (1 - e^2)}{(1 - e^2 \sin^2 \phi)^{\frac{3}{2}}}$ = radius of curvature in the meridian at ϕ

$\omega = \lambda - \lambda_0$

$\psi = \frac{\nu}{\rho}$ i.e. ratio of the radii of curvature at ϕ

$t = \tan \phi$

m = meridian distance from equator, computed using the following expression

$$m = a (A_0 \phi - A_2 \sin 2\phi + A_4 \sin 4\phi - A_6 \sin 6\phi)$$

where ϕ is in radians and

$$A_0 = 1 - \frac{e^2}{4} - \frac{3e^4}{64} - \frac{5e^6}{256}$$

$$A_2 = \frac{3}{8}\left(e^2 + \frac{e^4}{4} + \frac{15e^6}{128}\right)$$

$$A_4 = \frac{15}{256}\left(e^4 + \frac{3e^6}{4}\right)$$

$$A_6 = \frac{35e^6}{3072}$$

With the appropriate values for ellipsoids and scale factors, these formulae will work for any Transverse Mercator projection: UTM, SPCS, or whatever.

a = semi-major axis of the ellipsoid; a = 6,378,137 m for WGS84 (GRS80)

e^2 = eccentricity of the ellipsoid; e^2 = 0.006 694 3800 for WGS84.

The Reverse Problem

Here we wish to convert grid co-ordinates to geographicals. In order to convert the northing to latitude, we first need to calculate what is known as the foot-point latitude, ϕ', which is the latitude for which the meridian distance is equal to $\frac{N'}{k_0}$. This value can be calculated directly provided three other values, namely n, G and σ are calculated first. The choice of variable names for these three values is historical and isn't related to any other use of them.

$$n = \frac{a - b}{a + b}$$ where a and b are the semi-major and semi-minor axes

$$G = a\,(1 - n)\,(1 - n^2)\,(1 + \frac{9}{4}\,n^2 + \frac{225}{64}\,n^4)\,\frac{\pi}{180}$$

= mean length of an arc of one degree of the meridian

$$\sigma = \frac{m\ \pi}{180\ G} \qquad \text{use } m = \frac{N'}{k_0}$$

$$\begin{aligned}\phi' = \sigma &+ \left(\frac{3n}{2} - \frac{27n^3}{32}\right) \sin 2\sigma \\ &+ \left(\frac{21n^2}{16} - \frac{55n^4}{32}\right) \sin 4\sigma \\ &+ \left(\frac{151n^3}{96}\right) \sin 6\sigma \\ &+ \left(\frac{1097n^4}{512}\right) \sin 8\sigma\end{aligned}$$

With these values we can calculate the geographical co-ordinates directly. Note that t', ψ', ρ' and ν' are functions of the foot-point latitude and using the same formulae as listed above.

Latitude (in radians)

Let $$x = \frac{E'}{k_0\ \nu'}$$

$$\begin{aligned}\phi = \phi' &- \frac{t'}{k_0\ \rho'}\ \ x\ \frac{E'}{2} \\ &+ \frac{t'}{k_0\ \rho'}\ \frac{x^3\ E'}{24}\ [-4\ \psi'^2 + 9\ \psi'\ (1 - t'^2) + 12\ t'^2] \\ &- \frac{t'}{k_0\ \rho'}\ \frac{x^5\ E'}{720}\ [8\ \psi'^4\ (11 - 24\ t'^2) - 12\ \psi'^3\ (21 - 71\ t'^2) + 15\ \psi'^2\ (15 - 98\ t'^2 + 15\ t'^4) \\ &\qquad + 180\ \psi'\ (5\ t'^2 - 3\ t'^4) + 360\ t'^4] \\ &+ \frac{t'}{k_0\ \rho'}\ \frac{x^7\ E'}{40320}\ (1385 + 3633\ t'^2 + 4095\ t'^4 + 1575\ t'^6)\end{aligned}$$

Longitude (in radians)

Let $x = \frac{E'}{k_0 \nu'}$

$$\omega = \sec\phi' \; x$$
$$- \sec\phi' \frac{x^3}{6} (\psi' + 2 t'^2)$$
$$+ \sec\phi' \frac{x^5}{120} [-4 \psi'^3 (1 - 6 t'^2) + \psi'^2 (9 - 68 t'^2) + 72 \psi' t'^2 + 24 t'^4]$$
$$- \sec\phi' \frac{x^7}{5040} (61 + 662 t'^2 + 1320 t'^4 + 720 t'^6)$$

Grid Convergence (in radians)

Let $x = \frac{E'}{k_0 \nu'}$

$$\gamma = - t' x$$
$$+ t' \frac{x^3}{3} (-2 \psi'^2 + 3 \psi' + t'^2)$$
$$- t' \frac{x^5}{15} [\psi'^4 (11 - 24 t'^2) - 3 \psi'^3 (8 - 23 t'^2) + 5 \psi'^2 (3 - 14 t'^2) + 30 \psi' t'^2 + 3 t'^4]$$
$$+ t' \frac{x^7}{315} (17 + 77 t'^2 + 105 t'^4 + 45 t'^6)$$

Point Scale Factor

Let $x = \frac{E'^2}{k_0^2 \nu' \rho'}$

$$k = k_0 \left(1 + \frac{x}{2} + \frac{x^2}{24} \left(4 \psi' (1 - 6 t'^2) - 3 (1 - 16 t'^2) - \frac{24 t'^2}{\psi'} \right) + \frac{x^3}{720} \right)$$

Grid Convergence

We can compute the grid bearing from the true or geodetic azimuth of the line. The true azimuth can be determined from astronomical observations, for example. We use the grid convergence as follows:

Grid Bearing = Geodetic Azimuth + Grid Convergence

$$\beta = \alpha + \gamma$$

Thus grid convergence depends the distance of the point from the central meridian, which is a function of longitude (actually difference in longitude from the central meridian), adjusted by the latitude to allow for meridian convergence.

For longer lines, the grid convergence will change from one end of the line to the other, so we must compute a correction for this. This is the arc-to-chord correction, as the geodetic line appears as a curve on the projection. The arc-to-chord correction, δ, is presented below, but the change in azimuth of a line from one end to the other is expressed as follows:

$$\text{Bg at 2} = \text{Bg at 1} \pm 180° - \delta_{12} - \delta_{21}$$

Working on the Transverse Mercator Projection

With the Transverse Mercator projection SPCS Zones, many of the original expressions required radii of curvature of the ellipsoid at various points. For high-precision work, it is therefore necessary to know the latitude of where you are. But for most normal work, approximations can be made which will work quite well.

Point Scale Factor

It is often necessary to get the point scale factor at a point so that you can make a guess at a line scale factor to compute the next point's co-ordinates. The rigorous point scale factor in Redfearn's formulae can be reduced to the following:

$$k = k_0 \left(1 + \left(\frac{E'^2}{2\, r_m^2}\right) + \left(\frac{E'^4}{24\, r_m^4}\right)\right)$$

r_m is equal to $k_0^2\, \rho\, \nu$ at the mid-point of the line. You need the latitude to determine this accurately. However, the above formula is good to 1 part in 10 million, and good to 2 parts in 10 million if you omit the final term. We can simplify this even further.

$$k = k_0 + 1{\cdot}23\, E'^2\ 10^{-14} \qquad \text{(for meters)}$$

The above formula assumes that you are working in meters. Otherwise the following is required.

$$k = k_0 + 1{\cdot}32\, X'^2\ 10^{-13} \qquad \text{(for feet)}$$

The accuracy of this formula is good to 8 ppm at the equator, and better than 4 ppm in US latitudes.

Line Scale Factor

The simplified formula requires knowing your latitude, as it is expressed by the following.

$$K = k_0 \left(1 + \frac{(E_1'^2 + E_1'\, E_2' + E_2'^2)}{6\, r_m^2}\right)$$

However, you can get a result good to 1 ppm in any line up to 33 km in Easting by taking the point scale factor of the Easting of the mid-point of the line. For the slightly lower accuracy of 1 ppm up to 16 km of Easting, take the mean of the point scale factors at each end of the line. This is for UTM; you will get better results again for SPCS Zones.

Arc-to-Chord Correction

We can simplify the rigorous arc-to-chord correction to the following:

$$\delta_{12}'' = -\delta_{21}'' = -(N_2 - N_1)\ (2E_1' + E_2')\ \ 8.466 \times 10^{-10} \qquad \text{(for meters)}$$

assuming that we are using meters. If we are using feet, the correction becomes:

$$\delta_{12}'' = -\delta_{21}'' = -(Y_2 - Y_1)\ (2X_1' + X_2')\ \ 7{\cdot}86 \times 10^{-11} \qquad \text{(for feet)}$$

These corrections are good to better than 1" on a line over 50 km long on the zone boundary of a UTM zone. They will be better in a SPCS zone. They may cause problems when crossing the central meridian, owing to the line curving in an S-shape, in which case compute the δ values separately, i.e., use the formula for each point, rather then just reversing the sign.

The full, rigorous version of the arc-to-chord correction is:

$$\delta_{12}'' = -(N_2 - N_1)(2E_1' + E_2')\left[\frac{1 - (2E_1' + E_2')^2}{27r^2}\right]\frac{1}{6r^2 \sin 1''}$$

We can omit the part in the squared brackets without any worry. The effect on UTM for a 100 km north-south line on the UTM zone boundary (a worst case) is 0.08". So it won't be noticeable in a SPCS zone. If you are worried about it, you should be using the log-line formulae anyway, not the plane approximations!

$1/6r^2$ takes the values of 4.12784×10^{-15} at 0° latitude through to 4.09009×10^{-15} at 55°. Combining this with the cosec 1" (to convert radians to seconds), the value we get ranges from 8.514×10^{-10} at 0°, through 8.473×10^{-10} at 40°, to 8.436×10^{-10} at 55°. So an adopted value of 8.466×10^{-10} for SPCS zones in general is a reasonable compromise.

Grid Convergence

The simplified formulae for grid convergence:

$$\tan\gamma = -\sin\phi\ \tan\omega \qquad\qquad \omega = \lambda - \lambda_0$$

is good to better than 0·"1 on very long lines. As a consequence, the required values for ϕ and ω can be scaled from any good quality map for results sufficient for most purposes. If greater precision is required, use Redfearn's Formulae to compute the grid convergence.

Grid convergence is only required when converting grid bearing to true azimuths and *vice versa*.

Software Availability

Tucked away on a server is a MS-DOS-based piece of software that will convert between geographicals and grid co-ordinates (both ways) for Transverse Mercator, at least as far as UTM is concerned (it makes the assumption that $k_0 = 0{\cdot}9996$). It will work with nearly all the ellipsoids in common use (some 27 or so). You can find it on the Region 1 server at `I:\CLASS\GS521\CONVERT.EXE`. This package will probably be modified in future to allow it to work with SPCS TM zones.

Datum shift software based on Molodensky's method (such as that in the Idrisi GIS package) is sufficient for shifting map-based data (which is good to perhaps a couple of meters in ground co-ordinates), but not good enough for survey-based data, which is often internally consistent to millimeters. The Molodensky method has the advantage of simplicity in parameters and computation, and so is quite fast and well suited to converting large amounts of data from any particular datum to any other (given that you known the transformation parameters, of course), but bear in mind its limitations. A list of the Molodensky parameters for a goodly collection of datums was handed out in a previous class.

For conversions between NAD27 and NAD83, software is available from two sources for free.

NGS's NADCON (Version 2.10) is available from NOAA, NGS, Silver Springs, Maryland, or from their web site at: `http://www.ngs.noaa.gov/` There is a lot of other interesting software available there, as well, all for free. NADCON can also be found (for the CONUS area only) on the Region 1 server at the following location: `H:\CLASS\GS521\NADCON\NADCON210.EXE`. Read the `read.me` file there before using this software.

The US Army Topographic Engineering Center has CORPSCON (Version 3.01) available from the same site, although you have to skip over to their own site from NGS's.

Also available on the Region 1 server is a package for converting between E, N and ϕ, λ for SPCS 83 zones. You can find this at `H:\CLASS\GS521\SPCS_83\`. The file is `spcs_83.exe`, and the `spcs_83.doc` file explains what it is all about.

Four packages for doing forward and reverse geodetic line computations, in 2-D on the ellipsoid, and in 3-D with the ellipsoidal heights included, can be found at `H:\CLASS\GS521\GEOD_COMPS\`. See the `read.me` file for more details about these applications.

At the NGS site, you can also download software for converting between X, Y and ϕ, λ for SPCS 27 zones. However, this works in batch mode only, and requires the input data files to be in the Blue Book format. If you want to download this package, you will also need to look at some other NGS utilities for setting up and checking files in the Blue Book format.

Note that all these packages run in DOS mode, and are text-based. There are some Windows-based packages starting to appear from NGS, but they are still fairly few and far between.

Each package has specific applications, which it is wise to bear in mind during use. It might be a good idea to investigate the methods of transformation that each package applies to the data, so that you know what is going on.

Software for geoidal computations of GEOID99's N values can be found on the web. You can enter latitude and longitude directly to a web page, and the value of N will be returned directly. This is a quick and easy way to get N for any location in the conterminous US. The web page can be found at the following address:

`http://www.ngs.noaa.gov/GEOID/geoid_comp.html`