# GS521 Geodetic Control Surveying

## The State Plane Co-ordinate System

## Introduction

We will now consider the particular projection that is used for the Ohio Co-ordinate System, the Lambert Conformal Conic. It goes under a couple of different names, but this is the projection that is used in Ohio for plane co-ordinates.

Note that after December 31, 1999, you can only use the SPCS 1983 co-ordinates in Ohio, and these can only be in meters. So, if you want to use plane co-ordinates on the Ohio system, you are now legally obliged to metricate.

## Lambert's Conformal Conic Projection

In order to obtain grid co-ordinates on a Lambert projection, we must remember that the grid co-ordinate system is a rectangular system, which is different to the 'fan-shaped' appearance of the projected region. So there will be a grid convergence factor to be allowed for when undertaking computations in grid-co-ordinates alone.

The process is to convert the geographical co-ordinates to polar co-ordinates $(r, \theta)$, then convert these to rectangular grid co-ordinates. Consider the situation in the Northern Hemisphere, using the following diagram.
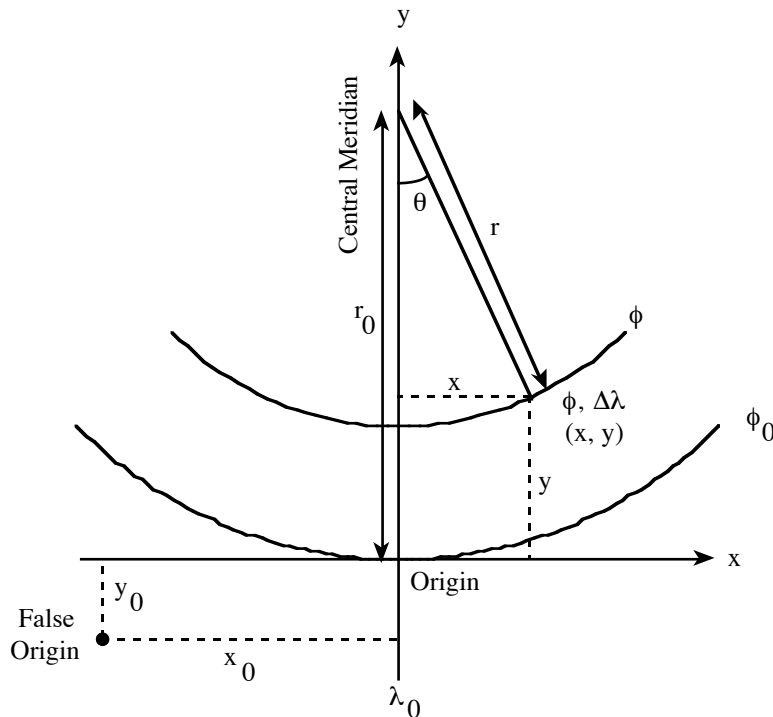


Here we have also placed a false origin, so that all co-ordinates on the grid will be positive. As this is a simple additive (and arbitrary) transformation, we can leave it until the last step.

In the diagram, r = radius of some parallel of latitude $\phi$; $r_0$ = radius of the parallel, $\phi_0$, upon which the true origin of the co-ordinate system is situated; (x, y) are the grid co-ordinates of the geographical point ($\phi$, $\Delta\lambda$); $\theta$ = the projection angle for a departure of $\Delta\lambda$ from the central meridian, which has a longitude of $\lambda_0$.

Note that we should ensure that $\Delta\lambda$ is positive when the point is east of the central meridian, which means that the correct determination for $\Delta\lambda$ is: $\Delta\lambda = \lambda - \lambda_0$.

If we have determined r and $\theta$ for the point ($\phi$, $\lambda$), as well as n, $r_0$ and c as discussed in the previous notes, we can then compute the grid co-ordinates as follows

$$x = r \sin \theta$$

$$y = r_0 - r \cos \theta$$

If we are using a false origin, we can then compute the final co-ordinates, X and Y, thus:

$$X = x_0 + x$$

$$Y = y_0 + y$$

and the task is complete. In many cases, of course, $y_0 = 0$, and $r_0$ is set sufficiently large to ensure that the entire area of interest in the map will have positive Y co-ordinates.

## Some Problems

Having developed the above, as well as the reverse case of grid to geographical co-ordinates, I then developed a spreadsheet to test the formulae in comparison to the values in the published tables for the SPCS Lambert zones. Computations derived from the above formulae and their direct reverse formulae did not agree with the computations made for the SPCS Lambert zones. In order to develop a set of formulae for Lambert's Conformal Conic that agree with the published tables, I have taken the formulae from the USGS publication by Snyder, *Map Projections – A Working Manual,* and tested them to ensure that they work properly. The following is the USGS formulae, with certain provisos attached.

Note that the formulae above and in the previous notes will work with a Lambert Conformal Conic projection, but not the one that the USGS set up for SPCS.

## SPCS Formulae

The following is derived from Snyder's *Map Projections – A Working Manual*.

If we are working on the ellipsoid, which we really have to do for most cases and certainly for the SPCS, we need the following information in order to convert geographical co-ordinates into grid co-ordinates: a, $e^2$, $\phi_1$, $\phi_2$, $\phi_0$, $\lambda_0$, $\phi$ and $\lambda$. We know a and $e^2$ for the particular ellipsoid being used, and the fixed latitude and longitude values are already determined for every SPCS zone. The attached lists give this information.

(Note that the SPCS 27 has co-ordinates in US Survey feet, and uses the Clarke 1866 ellipsoid. The SPCS 83 has co-ordinates in meters, and uses the GRS80 spheroid, which effectively is the same as WGS-84. Some states have either the US Survey foot or the International foot as alternative distance units; check which one is in use in the state you are working in at any particular time. Note that there is a datum shift between the two systems as well, and that you cannot really do a direct linear shift between them.)

With the basic information, we can compute the co-ordinates as follows:

$$x = r \sin \theta$$

$$y = r_0 - r \cos \theta$$

where the following formulae are used. Note that subscripts 0, 1 or 2 imply the use of subscripts in the formulae for r, m and t at the appropriate places.

The example of $r_0$ is given below. The terms t, m and F are used only to simplify computation and do not imply any real (or readily apparent) physical quantity or value.

$$r = a F t^n$$

$$\theta = n (\lambda - \lambda_0)$$

$$r_0 = a F t_0^n$$

$$n = \frac{\ln m_1 - \ln m_2}{\ln t_1 - \ln t_2} \qquad \text{(constant of the projection or cone)}$$

$$m = \frac{\cos \phi}{\sqrt{1 - e^2 \sin^2 \phi}}$$

$$t = \frac{\tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right)}{\left(\frac{1 - e \sin \phi}{1 + e \sin \phi}\right)^{\frac{e}{2}}}$$

or

$$= \sqrt{\left(\left(\frac{1 - \sin \phi}{1 + \sin \phi}\right) \left(\frac{1 + e \sin \phi}{1 - e \sin \phi}\right)^e\right)}$$

$$F = \frac{m_1}{n \, t_1^n}$$

Note that n, F and $r_0$ are constants for a particular map or SPCS zone and only need to be computed once.

When computing with these values, you will get results slightly different from those in the published tables for SPCS 27 (although the Ohio ones are currently out of print). The discrepancy is of the order of 20-30 meters which is relatively small, considering the r distances coming from the pole, and the relative error in a SPCS zone is quite small. The reason for this is that for calculating convenience 70 years ago, when the tables were developed, the tables were, in effect, calculated using the following variant of t, i.e.

$$t = \tan\left(\frac{\pi}{2} - \frac{\phi_g}{2}\right)$$

where $\phi_g$ is the geocentric latitude, also able to be expressed by

$$\phi_g = \arctan((1 - e^2) \tan \phi)$$

t is actually the cotangent of half the co-latitude of the conformal latitude, $\chi$, which is derived by assuming that a sphere was used as a kind of substitute for the ellipsoid in some calculations (see Snyder for greater details of this, if you are interested). As it happens, the expansions for $\chi$ and $\phi_g$ are numerically very nearly the same. However, the small differences still exist in the SPCS 27 tables. There are other smaller differences caused by the slightly lower precision of older desk-top calculating machines, compared to modern equipment, and the adaptation of formulae to suit these machines. However, these discrepancies are pretty small.

As you can see from the attached tables, most of the SPCS Lambert zones adopt $\phi_0$ as having a Y value of zero. It is chosen so as to be well south of the limits of the zone. For most of the Lambert zones, the central meridian gets a value of 2,000,000 feet (SPCS 27), and a range of meter values for SPCS 83. See the attached tables for exact data.

## The Reverse Problem

Unlike the Transverse Mercator projection, where the forward and reverse co-ordinate conversions are pretty straight-forward, if long-winded, the solution to the conversion of grid co-ordinates to geographicals on the Lambert Conformal Conic is a rather more complex affair.

Given $a$, $e^2$, $\phi_1$, $\phi_2$, $\phi_0$, $\lambda_0$, x and y (with false origin removed), we can calculate n, F and $r_0$ from the USGS formulae given above. Then we have to undertake an iterative solution to the following equation, solving for $\phi$ with a rough estimate for $\phi$, substituting that value back into the equation and solving for a better value of $\phi$. When there is no significant change in $\phi$, we can stop the process. A good starting estimate for $\phi$ is $\phi = \frac{\pi}{2} - 2 \arctan t$. The formula that is being iterated is:

$$\phi = \frac{\pi}{2} - 2 \arctan\left(t \left(\frac{1 - e \sin \phi}{1 + e \sin \phi}\right)^{e/2}\right)$$

where

$$t = \left(\frac{r}{a F}\right)^{1/n}$$

$$r = \sqrt{x^2 + (r_0 - y)^2}$$

$$\lambda = \frac{\theta}{n} + \lambda_0$$

$$\theta = \arctan\left(\frac{x}{r_0 - y}\right)$$

To avoid iterations, you can use the following series expansion (note that Snyder gives some clues for faster implementation of this expansion on page 19 of his book).

$$\begin{aligned}
\phi = \chi \;+\; & \left(\frac{e^2}{2} + \frac{5e^4}{24} + \frac{e^6}{12} + \frac{13e^8}{360} + \ldots\right) \sin 2\chi \\
+\; & \left(\frac{7e^4}{48} + \frac{29e^6}{240} + \frac{811e^8}{11520} + \ldots\right) \sin 4\chi \\
+\; & \left(\frac{7e^6}{120} + \frac{81e^8}{1120} + \ldots\right) \sin 6\chi \;+\; \left(\frac{4279e^8}{161280} + \ldots\right) \sin 8\chi
\end{aligned}$$

where

$$\chi = \frac{\pi}{2} - 2 \arctan t$$

If you are using the tables for SPCS 27 and these were derived from geocentric latitudes, you can use the same formulae, except that t for calculations leading to n, F and $r_0$ is calculated using:

$$t = \tan\left(\frac{\pi}{2} - \frac{\phi_g}{2}\right)$$

where

$$\phi_g = \frac{\pi}{2} - 2 \arctan t$$

and the t in this equation is derived from $\quad t = \left(\frac{r}{a\,F}\right)^{1/n}$

With the books of tables now out of print, you will probably have to develop your own programs or spreadsheets for these computations. Chapter 11 of Moffitt and Bossler gives several examples of how to use the interpolation tables to convert between geographical and grid co-ordinates, and you can use these examples to check your computations.

## Computations on the SPCS

For both of the main projections, there are some things in common. The first is that if we are using grid co-ordinates and grid bearings and distances, we need to convert the geodetic angles and distances that we actually measure to their grid equivalent prior to further computation. This further computation is completely 'plane', i.e., you can use common plane trig. for the computations. To do this geodetic to grid conversion, we use two main computational tools, and one additional tool.

The two main tools are the scale factor, which converts sea-level or ellipsoidal distances into grid distances, and the arc-to-chord correction, which converts measured angles to grid angles, allowing for the bending of the lines as they come from the Earth to the grid. The additional tool is the grid convergence, used to convert a true azimuth to a grid bearing.

## Lambert Conformal Conic Projection Conversion Tools

With this projection, the scale factor is a function of latitude. It is constant along a parallel of latitude and the same in all directions from a point. Because most of the SPCS zones are placed in between the standard parallels, the scale factor is commonly less that 1·0.

# Using SPCS 27

*Scale Factor*

The scale factor, k, is set to 1·0 at the standard parallels. For any other point, it can be determined from the following equation:

$$k \; = \; \frac{n \; r}{\nu \cos \phi}$$

where  n is the constant of the cone or projection (determined solely from $\phi_1$ and $\phi_2$);
r is the plotting radius of the parallel at latitude $\phi$;
$\nu$ is the radius of the ellipsoid at latitude $\phi$; and
$\phi$ is the latitude of the point.

This does tend to mean that we need to know the latitude of the point, which is a significant inconvenience. You can get this value from the tables, by computing the value of r from the co-ordinates, but there does not seem to be any simple and direct method of solving this from co-ordinates alone, as there is for the Transverse Mercator. However, this is a nice little research project!

*Arc-to-Chord Correction*

The other value one needs to get is the arc-to-chord correction. This can be expressed using the following equation, where $X_1, Y_1$ and $X_2, Y_2$ are the grid co-ordinates of the ends of the line, and $Y_0$ is the Y co-ordinate of the 'central parallel, $\phi_0$' of the projection or zone where it crosses the central meridian.

(Note that $Y_0$ and $\phi_0$ must not be confused with the Y co-ordinate and latitude of the co-ordinate origin of the zone, or the false northing. The central parallel of the projection or zone is a parallel roughly halfway between the standard parallels. All 'straight lines' on the Earth are curved when projected onto the map grid (except north-south lines), and these lines all curve away from the central parallel. We use the Y co-ordinate of the central parallel where it crosses the central meridian as an approximation to the correct Y co-ordinate at  any particular location, as the amount of curvature depends upon how far the line is from the central parallel. This explains its presence in the arc-to-chord correction formula.

$R_0$ is the mean radius of curvature at $\phi_0$. $\phi_0$ is known for each zone, (it is in the tables) and so $R_0$ can be computed, so these constants need only be determined once for every zone.

You obtain $\phi_0$ with the following equation.

$$\phi_0 \; = \; \arcsin(n)$$

where n is the value derived above, i.e., the constant of the cone or projection. Then substitute this latitude into the formula to compute r, the plotting radius, ($r \; = \; a \, F \, t^n$) and from this point the $Y_0$ co-ordinate can easily be determined.

The arc-to-chord correction can be expressed as follows.

$$\delta_{12}" \; = \; \frac{X_2 - X_1}{2 \, R_0{}^2 \sin 1"} \left( Y_1 - Y_0 + \frac{Y_2 - Y_1}{3} \right)$$

This is the arc-to-chord correction at Point 1, looking along the line to point 2, expressed in seconds of arc, which is added to the measured (geodetic, arc) direction to get the grid (chord) direction. At Point 2 looking towards point 1, the correction will be as follows (also expressed in seconds of arc).

$$\delta_{21}" = \frac{X_1 - X_2}{2 R_0{}^2 \sin 1"} \left( Y_2 - Y_0 + \frac{Y_1 - Y_2}{3} \right)$$

The only difference is that the correction has the opposite sign, and that the terms have the subscripts reversed (it is the reversing of the X co-ordinates that results in the change in sign of the result. For most work, we can say that $\delta_{12} = - \delta_{21}$. It is only at the finer levels of precision over very long lines that we will have any significant difference, and these are unlikely to be computed into State Plane Co-ordinates.

The only point to be wary with the arc-to-chord correction occurs if the line crosses the central parallel, $\phi_0$. If this happens the right way (the line has to have the proportions on either side of the line about right), the line can project as 'S'-shaped, and the arc-to-chord corrections have the same sign at opposite ends of the line. You check for this by seeing if your Y co-ordinates for any line straddle the $Y_0$ value.

If you have a line that does cross the $Y_0$ line, work out the arc-to-chord corrections for each end of the line separately and they should produce the correct results. But note that this situation is not very common.

For most work on SPCS 27, where we will be working in US Survey Feet, we can reduce the constants in the above equations so that the following can be used:

$$\delta" = 2 \cdot 36 (X_2 - X_1) \left( Y_1 - Y_0 + \frac{Y_2 - Y_1}{3} \right) \times 10^{-10}$$

and it should be satisfactory to use it from both ends of the line.

If a line runs north–south, the differences in X co-ordinates are zero, and so is the correction.

*Grid Convergence*

The final thing you need to know is the grid convergence. This will allow you to convert a true azimuth to a grid bearing and *vice versa*. The grid convergence is $\theta$, which is a function of longitude difference from the central meridian.

$$\theta = n \Delta\lambda = n (\lambda - \lambda_0) = \arctan \left( \frac{x}{r_0 - y} \right)$$

It can be calculated directly from the co-ordinates of the point, after removing false origin effects. If you happen to know the longitude of the point, you can also compute it directly.

# The SPCS 83 Situation

The main change between the 1927 and 1983 SPCS is that the meter has been adopted for use, although it is also possible to use feet in some states; which kind of foot is to be used is determined by the individual state. There is also the change in ellipsoid, which leads to changes in the various constants for each projection. You need to make sure that you are using the correct set of constants.

With the tables produced by NGS, there were some significant changes in the way the tables were set up for computations between the 1927 and 1983 systems. If you are using the tables, make sure that you are up to speed with the changes. Note that the 1983 system can be computed directly from the formulae, as no short-cuts were used, owing to the improvements in computational technology since the 1930s. You can do all the computations you need for this on a decent pocket calculator.

Note that tables for both systems are out of print and hard to get, so you may be reduced to computation, anyway. With luck, you will only have to work on the 1983 system, and you experience with the 1927 system may be only when updating older work. Note that the OSU main library has a full set of the tables for the 1927 system, for all the states.

*Scale Factor*

The scale factor can be determined from tables or from a set of simple equations presented in the new set of tables. Each zone has a set of constants that are used to compute most of the required pieces of information.

*Arc-to-Chord Correction*

The only difference here is the notation. Change the Xs to Es and the Ys to Ns. If we are using meters, the following is the arc-to-chord correction with constants evaluated.

$$\delta_{12}'' \;=\; 25{\cdot}4\,(E_2 - E_1)\left(N_1 - N_0 + \frac{N_2 - N_1}{3}\right)\;\times\;10^{-10}$$

If you are using feet (of either kind) the correction can be written as follows.

$$\delta_{12}'' \;=\; 2{\cdot}36\,(E_2 - E_1)\left(N_1 - N_0 + \frac{N_2 - N_1}{3}\right)\;\times\;10^{-10}$$

$N_0$ has the same function as $Y_0$ in SPCS 27. Lines crossing $N_0$ may have the same reflex curve problem, which means you should check those kinds of lines.

*Grid Convergence*

This is computed in the same way as for SPCS 27.

## Line Scale Factor

The scale factors discussed above are all computed for a point (well, at a specific altitude, actually). But the scale factors at each end of a line will tend to differ. There are three solutions to determining the scale factor for a whole line. The first is to use the mean of the point scale factors at either end of the line. This should be accurate to about 1 ppm in lines with a 10 mile north-south component. The second is to use the scale factor of the mid-point of the line. This should be of about the same precision as the mean of each end of the line. Either of these will be good enough for general work.

For work of the highest precision, use the following formula to determine the line scale factor:

$$K = \frac{k_1 + 4k_m + k_2}{6}$$

where K is the line scale factor, $k_1$ and $k_2$ are the point scale factors at each end of the line and $k_m$ is the point scale factor at the mid-point of the line.

## Legal Matters

As far as Ohio is concerned, there are certain legal requirements for using the SPCS. The 1927 and 1983 Ohio Co-ordinate Systems are defined, established and legal requirements are laid out in Chapter 157 of the Ohio Revised Code. Everything is as in Snyder's book (and all others, of course), but there are a couple of important points to note.

While the 1927 system is specifically described as using US Survey Feet in ORC 157.03 (and decimals of a foot), the 1983 is specified as using metres (using the correct spelling!) and **nothing else**. You cannot use feet (of any sort) if you are working on the 1983 system, only metres. Of particular interest is section 157.11 of the ORC which states:

"In accordance with sections 157.01 to 157.10 of the Revised Code, the Ohio co-ordinate system of 1927 shall not be used after 1999 and the Ohio co-ordinate system of 1983 shall be used after that date."

While the section goes on to state that land descriptions prepared prior to 2000 using the 1927 system do not have to be converted to 1983, it means that all surveys that reference the Ohio co-ordinate system on or after 1st January, 2000, must be in metres. In effect, the metric system of measurement for surveys is now the legally required system. As the Board in Ohio is trying to move things to the metric system, in accordance with the ORC, and the professional registration examinations are being converted to metric data wherever possible, it is wise to be metric-aware.